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# Relativistic statistical thermodynamics of dense photon gas

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We discuss some aspects of interactions of high-frequency electromagnetic waves with plasmas, assuming that the intensity of radiation is sufficiently large, so that the photon-photon interaction is more likely than the photon-plasma particle interaction. In the stationary limit, solving the kinetic equation of the photon gas, we derive a distribution function. With this distribution function at hand, we investigate the adiabatic photon self-capture and obtain the number density of the trapped photons. We employ the distribution function to calculate the thermodynamic quantities for the photon gas. Having expressions of the entropy and the pressure of the photon gas, we define the heat capacities and exhibit the existence of the ratio of the specific heats $\Gamma$, which equals 7/6 for nonrelativistic temperatures. In addition, we disclose the magnitude of the mean square fluctuation of the number of photons. Finally, we discuss the uniform expansion of the photon gas.

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## I. INTRODUCTION

The recent development of astronomical observations has revealed that our universe is full of enigmatic explosive phenomena, such as jets, bursts, and flares. It is possible now to study extremely complex phenomena (supernova explosion, Gamma-ray bursts, etc.) of astrophysics in laboratories using intense and ultraintense lasers. Intense lasers have been used to investigate hydrodynamics, radiation flow, opacities, etc., related to supernova explosions, giant planets, and other astrophysical systems [1]. Thus the study of the properties of such radiation (strong and superstrong laser pulse, nonthermal equilibrium cosmic field radiation, etc.) is of vital importance. The development of compact, high-power, short pulse, efficient lasers is a fast moving technology. In the field of superstrong femtosecond pulses, it is expected that the character of the nonlinear response of the medium will radically change. Currently, lasers produce pulses whose intensity approaches $10^{22}$ W/cm$^2$ [2]. With a further increase of intensity [3], we may encounter novel physical processes, where the quantum electrodynamic description may be needed. Recently, the nonlinear collective effects in quantum electrodynamics has been reviewed in Ref. [4].

We have shown in Ref. [5], and later Medvedev in [6], where thermodynamic properties of a photon gas in electron-positron plasmas were studied (results of [6] were recalculated recently in [7]), that the behavior of photons in a plasma is radically different from that in a vacuum. Namely, plasma particles perform oscillatory motion in the field of electromagnetic (EM) waves affecting the radiation field. The oscillation of electrons in an isotropic homogeneous plasma leads to the index of refraction, which depends on the frequency of the radiation, and is not close to unity for a dense plasma, i.e.,

$$R^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}, \tag{1}$$

where $\omega_p = \left(\frac{4\pi e^2 n}{m_{0e}\gamma}\right)^{1/2}$ for an electron-ion plasma (neglecting the ion contribution) and $\omega_p = \left(2\frac{4\pi e^2 n}{m_{0e}\gamma}\right)^{1/2}$ for an electron-positron plasma ($-e$, $m_{0e}$, $n$, and $\gamma$ are electron charge, the rest mass, density, and the relativistic gamma factor of the electrons, respectively).

Rewriting Eq. (1) in terms of an energy $\varepsilon = \hbar\omega$ and momentum $p = \hbar k$ (where $\hbar$ is the Planck constant divided by $2\pi$) and introducing $m_\gamma = \hbar\omega_p/c^2$, we obtain the expression for the energy of a single photon

$$\varepsilon_\gamma = c(p_\gamma^2 + m_\gamma^2 c^2)^{1/2} = m_\gamma c^2 \left(1 - \frac{u_\gamma^2}{c^2}\right)^{-1/2}, \tag{2}$$

which is expressed through the standard formula for the velocity of energy transport

$$u_\gamma = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = \frac{\partial\omega}{\partial k}. \tag{3}$$

For the momentum of a photon we can write

$$\vec{p}_\gamma = \hbar\vec{k} = m_\gamma\gamma_\gamma\vec{u}_\gamma = m_\gamma\left(1 - \frac{u_\gamma^2}{c^2}\right)^{-1/2}\vec{u}_\gamma. \tag{4}$$

The form of Eq. (2) coincides with the expression for the total relativistic energy of massive particles, so that a rest mass $m_\gamma$ is associated with the photon in a plasma [5,6,8]. We note here that two important features of photons follow from Eq. (2). Namely, first at $p_\gamma = 0$, $\varepsilon_\gamma = m_\gamma c^2$ is not zero. Second, the rest mass of photons depends on the plasma

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density, or the volume as $m_\gamma = \frac{\hbar}{c^2}\left(\frac{4\pi e^2}{m_{0e}\gamma}\frac{N_e}{V}\right)^{1/2}$. In view of this analogy between a photon in a plasma and a free material particle, we can treat the photon gas in the plasma just as a subsystem of particles that have nonzero rest mass.

In the approximation of geometric optics we may work in terms of rays (photons) instead of waves. In a homogeneous isotropic medium the direction of the ray coincides with that of the normal of the wave surface. In practice, however, we often have to deal with pulses formed by a group of waves. For the wave packet, i.e., a quasimonochromatic group of waves, the Fourier component of wave energy is by definition very "sharp" and appreciably different from zero only in a narrow range of frequencies and wave vectors near the carrier $\omega$ and $k$ of the pulse. In plasmas, as follows from Eq. (1), the group velocity $u = \frac{\partial\omega}{\partial k} < c$. Thus the wave packets of light are propagated with a group velocity which is less than the speed of light, in accordance with the theory of relativity [9]. It is also well-known that the introduction of group velocity is valid in the case of weak field (for the linear waves). However, for strong nonlinear waves in plasmas a concept of group velocity is meaningless. In this case we should define the mean velocity of the group of photons taking into account their interaction with plasma particles. We note that the wide range of applicability of the approximation of geometrical optics is due to the fact that the properties of plasma usually vary slowly in space and time, i.e., the properties of the medium change very little over distances of the order of the wavelength (or of some characteristic length).

Let us recall some purely quantum mechanical features of a macroscopic system. It is well-known that there is an extremely high density of levels in the energy eigenvalue spectrum of a macroscopic system. We know also that the number of levels in a given finite range of the energy spectrum of a macroscopic system increases exponentially with the number of particles $N$ in the system, and separations between levels are given by numbers of the $10^{-N}$. Therefore we can conclude that in such a case the spectrum is almost continuous and a quasiclassical approximation is applicable. To support this statement, we will discuss some conditions which will allow us to use a quasiclassical approximation. We start from the uncertainty principle in the relativistic case [10] for photons. In the relativistic theory a coordinate uncertainty in a frame of reference in which the particle is moving with energy $\varepsilon$ is

$$\Delta q \sim \frac{c\hbar}{\varepsilon} = \frac{c}{\omega}. \tag{5}$$

Estimating this quantity for the isotropic plasma, we obtain for the underdense plasma, $\triangle q \sim \lambda$ ($\lambda$ is the wavelength), and for the overdense plasma $\triangle q \sim c/\omega_p$. This means that the coordinates of the photon are meaningful only in those cases where the characteristic dimensions of the problem are large in comparison with the wavelength or the anomalous skin depth.

We now consider the quantization of an EM field. In the quantum field theory, the Hamiltonian has the same form as in classical field theory, the only difference is that now $E$ and $B$ are operators, i.e.,

$$\vec{H}_f = \frac{1}{8\pi}\int (E^2 + B^2)d\vec{r} \tag{6}$$

and the eigenvalues of this Hamiltonian are

$$H = \sum_{\mathbf{k},\sigma}\left(n_{\mathbf{k},\sigma} + \frac{1}{2}\right)\hbar\omega(k), \tag{7}$$

where the occupation numbers $n_{\mathbf{k},\sigma}$ are integers, and $\sigma$ stands for the polarization.

The eigenvalues of the momentum operator are

$$\vec{p} = \sum_{\mathbf{k},\sigma}\left(n_{\mathbf{k},\sigma} + \frac{1}{2}\right)\hbar\vec{k}. \tag{8}$$

The expressions (7) and (8) enable one to introduce the concept of photons, i.e., the EM field as an ensemble of particles each with energy $\hbar\omega$ and momentum $\hbar\vec{k}$. The occupation numbers $n_{\mathbf{k},\sigma}$ now represent the numbers of photons with given $\vec{k}$ and polarization $\sigma$.

The properties of a photon gas are known to be similar to the classical properties when the photon numbers $n_{\mathbf{k},\sigma}$ are large. This statement allows us to define the condition for a value of an amplitude of the electric field, which indicates the validity of the classical approach of the photon gas. To this end, we shall estimate the total field energy per unit volume, which is proportional to $|E|^2$. In the quasiclassical limit, the total number of proper oscillations with the magnitude of the wave vector in the interval $dk$ is

$$\frac{Vk^2dk}{\pi^2} = \frac{V\omega^2}{\pi^2c^3}R^2\frac{d}{d\omega}(\omega R)d\omega. \tag{9}$$

Noting Eq. (1) for an isotropic plasma expression (9) reduces to

$$\frac{V\omega^2}{\pi^2c^3}Rd\omega. \tag{10}$$

For the energy density of the field we have

$$|E|^2 = \int \hbar\omega n(\omega)\frac{\omega^2 Rd\omega}{\pi^2c^3} \sim \frac{R\hbar}{\pi^2c^3}\omega^4 n(\omega). \tag{11}$$

As we have mentioned above there is a similarity between the quantum and the classical system, provided $n_\omega \gg 1$, i.e., when

$$|E| \gg (\hbar cR)^{1/2}\left(\frac{\omega}{c}\right)^2. \tag{12}$$

From this it is clear that for the static field, i.e., $\omega=0$, $|E|$ is always classical. The same situation occurs for the overdense plasma, as $R\to 0$. In general, a high-frequency EM field, if sufficiently weak, can never be quasiclassical. Thus the inequality (12) is the required condition, which allows the EM field to be treated as quasiclassical.

## II. FIRST LAW OF RELATIVISTIC THERMODYNAMICS

We now consider a system which is a dilute gas composed of electrons, ions, and photons ($e-i-\gamma$), or electrons, positrons, and photons ($e-p-\gamma$), and describe this compressible and continuous medium in terms of its macroscopic properties such as entropy, pressure, density, temperature, etc.

First, we calculate the thermodynamical quantities developing the statistical mechanics in the presence of a strong EM field. It was shown in Ref. [11] that in the case of the relativistically intense (circularly polarized) EM waves propagation into a plasma, the momentum $e_\alpha \vec{A}_\perp / c$ ($\vec{A}_\perp$ is the perpendicular component of the vector potential of the EM waves, $\alpha$ stands for the particle species) can be much larger than the perpendicular components of the thermal momentum of the particles. Hence the perpendicular momentum of particles is just $\vec{p}_{\perp\alpha} = -\frac{e_\alpha \vec{A}_\perp}{c}$, whereas the momentum of particles along the propagation of EM waves remains thermal. In the following, we study a closed system for a period of time that is long compared with its relaxation time. This implies that the system is in complete statistical equilibrium.

Introducing $E$ as the internal energy in a volume $V$ of the three component gas, the first law of thermodynamics reads (index $t$ stands for total)

$$dE_t = dQ_t - P_t dV, \tag{13}$$

where $P_t$ is the total pressure, or

$$P_t = P_e + P_{i(p)} + P_\gamma. \tag{14}$$

In the case when a plasma is in a superstrong EM field, the pressure becomes anisotropic. For instance, in the case of a relativistically intense circularly polarized EM field the total pressure is written as

$$P_t = \sum_\alpha (P_{\perp\alpha} + P_{\parallel\alpha}) + P_\gamma, \tag{15}$$

where

$$P_{\perp\alpha} = \frac{2}{3} \frac{n_\alpha m_{0\alpha} c^2 a_\alpha^2}{\sqrt{1+a_\alpha^2}} \frac{K_0(\beta_\alpha \sqrt{1+a_\alpha^2})}{K_1(\beta_\alpha \sqrt{1+a_\alpha^2})}, \tag{16}$$

$$P_{\parallel\alpha} = \frac{1}{3} n_\alpha T_\alpha. \tag{17}$$

Here $n_\alpha = \frac{n_{0\alpha}\sqrt{1+a_\alpha^2}}{K_1(\beta_\alpha)} K_1(\beta_\alpha \sqrt{1+a_\alpha^2})$ is the density of particles, $a_\alpha = \frac{e_\alpha \vec{A}_\perp}{m_{0\alpha} c^2}$, $\beta_\alpha = \frac{m_{0\alpha} c^2}{T_\alpha}$, and $K_\ell(X)$ is the McDonald function of $\ell$ order.

Deriving expressions (16) and (17) use was made of the distribution function

$$f_\alpha = B \delta\left(\vec{p}_{\perp\alpha} + \frac{e_\alpha}{c}\vec{A}_\perp\right) \exp\left\{ - \frac{c\sqrt{m_{0\alpha}^2 c^2 + p_{\perp\alpha}^2 + p_{\parallel\alpha}^2}}{T_\alpha} \right\}, \tag{18}$$

where $B$ is the normalization constant and $\delta(x)$ is Dirac's function. If we integrate expression (18) over $\vec{p}_\perp$, we obtain the distribution function, which was derived in [11], i.e.,

$$f_\alpha(p_{\parallel\alpha}, a_\alpha^2) = \int d\vec{p}_\perp f_\alpha(p_\parallel, \vec{p}_\perp, a_\alpha^2) = \frac{n_{0\alpha}}{m_{0\alpha} c} \frac{1}{K_1(\beta_\alpha)} \exp\{-\beta_\alpha \sqrt{1 + a_\alpha^2 + p_{\parallel\alpha}^2/(m_{0\alpha}^2 c^2)}\}. \tag{19}$$

We note here that distribution functions (18) and (19) give a complete description of the microscopic properties of the gas in the presence of superstrong radiation.

In Eq. (13) the $dQ_t$ is the amount of heat that is gained or lost by the system, which has the form

$$dQ_t = T_e dS_e + T_{i(p)} dS_{i(p)} + T_\gamma dS_\gamma, \tag{20}$$

where

$$S_\alpha = -V \int dp_\parallel \int d\vec{p}_\perp f_\alpha \ln f_\alpha \tag{21}$$

is the entropy of the particles.

Introducing the entropy per particle and using expression (19), we obtain

$$\frac{S_\alpha}{N} = -\frac{1}{n_\alpha} \int dp_\parallel f_\alpha(p_{\parallel\alpha}, a_\alpha^2) \ln f_\alpha(p_{\parallel\alpha}, a_\alpha^2). \tag{22}$$

After substitution of $f_\alpha(p_{\parallel\alpha}, a_\alpha^2)$ into Eq. (22), a simple integration over $p_\parallel$ gives

$$\frac{S_\alpha}{N} = -\left[ \ln \frac{n_\alpha}{m_{0\alpha} c K_1(\beta_\alpha)} + 1 - \beta_\alpha \sqrt{1+a_\alpha^2} \frac{K_2(\beta_\alpha \sqrt{1+a_\alpha^2})}{K_1(\beta_\alpha \sqrt{1+a_\alpha^2})} \right]. \tag{23}$$

In order to calculate the pressure and the entropy of the photon gas, we use the Bose distribution function [5]. The result is

$$P_\gamma = \frac{T_\gamma^4 \beta_\gamma^2}{\pi^2 (\hbar c)^3} \sum_{\ell=1}^{\infty} \frac{e^{\ell\beta_\gamma}}{\ell^2} K_2(\ell\beta_\gamma) \tag{24}$$

and

$$S_\gamma = \frac{V T_\gamma^3 \beta_\gamma^2}{\pi^2 (\hbar c)^3} \sum_{\ell=1}^{\infty} \frac{e^{\ell\beta_\gamma}}{\ell^2} \left[ \ell\beta_\gamma \left(1 - \frac{\ell\beta_\gamma}{4}\right) K_3(\ell\beta_\gamma) + \frac{\ell^2 \beta_\gamma^2}{4} K_1(\ell\beta_\gamma) \right], \tag{25}$$

where $\beta_\gamma = \frac{m_\gamma c^2}{T_\gamma} = \frac{\hbar}{T_\gamma}\left(\frac{4\pi e^2}{m_{0e}\gamma} \frac{N_e}{V}\right)^{1/2}$.

We now suppose that in each subsystem the entropy is conserved, i.e., $S_e$, $S_{i(p)}$, and $S_\gamma$ are constant. We note here that the relaxation in a photon-plasma system is a two-stage process. First, the statistical equilibrium is established in each subsystem independently, at first in a plasma, since photons usually have much longer mean free paths than charged particles, and then in a photon gas. Slower processes of the equalization of the photon and the plasma temperatures will take place afterwards. Since for an adiabatic process $dS \equiv 0$, we obtain the adiabatic equation for material particles from Eq. (23),

$$\frac{n_\alpha}{K_1(\beta_\alpha)}\exp\{-\beta_\alpha\sqrt{1+a_\alpha^2}G\}=\text{const}, \tag{26}$$

where

$$G=\frac{K_2(\beta_\alpha\sqrt{1+a_\alpha^2})}{K_1(\beta_\alpha\sqrt{1+a_\alpha^2})}.$$

For clarity, we consider three cases. First, for the relativistic temperatures $\beta_\alpha\sqrt{1+a_\alpha^2}\ll 1$, we get

$$\frac{n_\alpha}{T_\alpha}\left(1+\frac{e_\alpha^2A_\perp^2}{T_\alpha^2}\right)=\text{const}. \tag{27}$$

This expression shows that the thermal kinetic energy due to the thermal motion of particles along the propagation of EM waves, $T_\alpha=T_{\parallel\alpha}$, dominates the energy of the waves, and the second term in the bracket in Eq. (27) is less than unity. Hence we can neglect the second term in the bracket to obtain

$$VT_\alpha=V^{\Gamma-1}T_\alpha=\text{const},$$

from which follows the expression for the ratio of the specific heats

$$\Gamma=\frac{C_P}{C_V}=2.$$

In the opposite limit, that is for the nonrelativistic temperatures $\beta_\alpha\sqrt{1+a_\alpha^2}\gg 1$, we obtain

$$\frac{n_\alpha}{T_\alpha^{1/2}}\exp\{\beta_\alpha(1-\sqrt{1+a_\alpha^2})\}=\text{const}. \tag{28}$$

Finally, in the case when the temperature is ultrarelativistic, $T_\alpha\gg m_{0\alpha}c^2$, and also the radiation, i.e., $a_\alpha^2\gg 1$, then $\beta_\alpha\sqrt{1+a_\alpha^2}\approx\frac{e_\alpha|A_\perp|}{T_\alpha}$ and can be of the order of unity. The adiabatic equation now reads

$$\frac{n_\alpha}{T_\alpha^{1/2}}\exp\left\{-\frac{e_\alpha|A_\perp|}{T_\alpha}\right\}=\text{const}. \tag{29}$$

For the subsystem of photons, the asymptotic behavior of Eq. (25) for $\beta_\gamma\ll 1$ leads to

$$S_\gamma=S_{0\gamma}(1+0.83\beta_\gamma), \tag{30}$$

where the second term is due to the mass of the photon, and $S_{0\gamma}=\frac{4\pi^2}{45}\left(\frac{T_\gamma}{\hbar c}\right)^3V$ is the entropy of the photon gas in vacuum.

For the case $\beta_\gamma\gg 1$ Eq. (25) becomes

$$S_\gamma=S_{0\gamma}0.48\beta_\gamma^{3/2}. \tag{31}$$

In this case the entropy depends on the temperature and the volume as follows:

$$S_\gamma\sim T_\gamma^{3/2}V^{1/4}. \tag{32}$$

Thus, for the adiabatic process, we obtain

$$T_\gamma V^{1/6}=T_\gamma V^{\Gamma-1}=\text{const}. \tag{33}$$

We specifically emphasize that in contrast to the vacuum case, we can here define the ratio of the specific heats for the photon gas, and in the case of nonrelativistic temperatures the ratio of the specific heats for the photon gas is $\Gamma=\frac{7}{6}$.

As we have indicated in the Introduction, the nature of photons in plasmas is quit different from the one in vacuum. In plasma the photon has a rest mass that depends on the volume, and hence we can write for the mean square fluctuation of the number of photons

$$\langle(\Delta N_\gamma)^2\rangle=-\frac{T_\gamma N_\gamma^3}{V^2}\left(\frac{\partial V}{\partial P_\gamma}\right)_{T_\gamma}. \tag{34}$$

The derivation of this equation is well-known [12]. The limitations on its validity were pointed out, and a discussion on the mean square relative fluctuation in number of particles for an ideal relativistic Bose gas was reported by Dunning-Davies [13].

We now examine fluctuations in the distribution of photons over the various "quantum" states. Let $n_K$ be their occupation numbers in the $K$th quantum state. The mean values $\langle n_K\rangle=n_\gamma$ of these numbers are

$$n_\gamma=\frac{1}{\exp\left\{\dfrac{\varepsilon(K)-\mu_\gamma}{T_\gamma}\right\}-1}. \tag{35}$$

Recalling Eq. (34), we get

$$\langle(\Delta n_K)^2\rangle=T_\gamma\frac{\partial n_\gamma}{\partial\mu} \tag{36}$$

or

$$\langle(\Delta n_K)^2\rangle=n_\gamma(1+n_\gamma). \tag{37}$$

It is important to emphasize that in Eq. (37) the first term reflects the corpuscular behavior of the photons, whereas the second term is of wave origin. More precisely, it is the result of the irregular interference of EM waves. One can see from Eq. (37) that in the case when $|\varepsilon(K)-\mu_\gamma|\gg T_\gamma$, the first term is larger than the second one. This implies that photons are neutral particles. In the opposite case $|\varepsilon(K)-\mu_\gamma|\ll T_\gamma$, i.e., for the classical approach of fluctuation of EM waves, Eq. (37) exhibits that the relative fluctuations of the number of photons does not decrease, when the mean number of photons increases, so that

$$\frac{\langle(\Delta n_K)^2\rangle}{n_\gamma^2}\sim 1.$$

Thus we may conclude that in the range $|\varepsilon(K)-\mu_\gamma|\gg T_\gamma$, i.e., $n_\gamma\ll 1$, the radiation resembles the ideal gas of the particles-photons, and in the range $|\varepsilon(K)-\mu_\gamma|\ll T_\gamma$, i.e., $n_\gamma\gg 1$, the radiation represents the system of classical electromagnetic waves.

## III. BOLTZMANN H-THEOREM FOR PHOTON GAS

Recently in Ref. [14] a new version of the Pauli equation for the photon gas was derived from a general kinetic equation (which is of the type of the Wigner-Moyal equation [15]) for the EM spectral intensity [16–18]. In the limit of the spatial homogeneity for the distribution function the Pauli equation reads

$$\frac{\partial N(\vec{k},t)}{\partial t} = \sum_{\pm} \int \frac{d^3k'}{(2\pi)^3} W_{\pm}(\vec{k}',\vec{k}) \times \left[ \frac{\omega(\vec{k})}{\omega(\vec{k}')} N(\vec{k}',t) - N(\vec{k},t) \right]. \qquad (38)$$

Here $W_{\pm}(\vec{k}',\vec{k})$ is the scattering rate

$$W_{\pm}(\vec{k}',\vec{k}) = \frac{\pi}{4} \frac{\omega_p^4 |\delta\rho(q)|^2}{\omega(\vec{k} \pm \vec{q}/2)\omega(\vec{k})} \delta(\Omega - \vec{q}\vec{u}_{\pm}), \qquad (39)$$

where $\vec{k}'=\vec{k}+\vec{q}$, $\vec{u}_{\pm}=\frac{(\vec{k}\pm\vec{q}/2)c^2}{\omega(\vec{k}\pm\vec{q}/2)}$, $\omega(\vec{k})$, $\vec{k}$ and $\Omega$, $\vec{q}$ are the frequencies, wave vectors of the transverse and longitudinal photons (photonikos), respectively, and $N(\vec{k},t)$ is the distribution function of photons.

We now discuss some implications of Eq. (38). Namely, this equation exhibits the irreversible processes, and is the mathematical basis for a H-theorem. The relaxation process is accompanied by an increase in the entropy of the photon gas. Note that the equation type of Eq. (38) has been obtained for the first time by Pauli for a quantum system and applied to study of irreversible processes [19]. Later Van Hove [20], Prigogine [21], and Chester [22] developed a general theory of irreversible processes. Namely, it was shown by them that the statistical equilibrium of the system is triggered by a small perturbation in potential energy, and the probability of the transition $(\vec{k},\vec{k}')$ can be calculated by the first order approximation of the nonstationary theory of perturbation.

Equation (38), derived for a dense photon gas, is pure classical and describes the three wave interaction. Namely, the photon passing through the photon bunch absorbs and emits photonikos, with frequencies $\Omega = \mp(\omega - \omega')$ and wave vectors $\vec{q} = \mp(\vec{k}-\vec{k}')$. The integral in Eq. (38) is the elastic collision integral and describes the photon scattering process on the variation of shape of the photon bunch. This equation indicates that the equilibrium of the photon gas is triggered by the perturbation $\delta\rho = \delta(n_i n_o \gamma)$.

In the limit of spatial homogeneity and quasiclassical approximation we can define the entropy of a photon gas as

$$S = -K_B V \int \frac{d^3k}{4\pi^3} \{ N(\vec{k},t) \ln N(\vec{k},t) - [N(\vec{k},t)+1] \ln(1+N) \}, \qquad (40)$$

where $K_B$ is Boltzmann's constant.

Differentiating this expression with respect to time, we obtain

$$\frac{dS}{dt} = VK_B \int \frac{d^3k}{4\pi^3} \ln\left( \frac{1+N(\vec{k},t)}{N(\vec{k},t)} \right) \frac{\partial N(\vec{k},t)}{\partial t}. \qquad (41)$$

From Eq. (38), where we take $\frac{\omega(\vec{k})}{\omega(\vec{k}')} \approx 1$ since we consider the case, when the wave number $q$ of the photoniko is much less than the wave number $k$ of the photons, we substitute $\frac{\partial N(k,t)}{\partial t}$ into Eq. (41) to obtain

$$\frac{dS}{dt} = VK_B \sum_{\pm} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \ln\left( \frac{1+N(\vec{k},t)}{N(\vec{k},t)} \right) W_{\pm}(\vec{k}',\vec{k}) \times [N(\vec{k}',t) - N(\vec{k},t)]. \qquad (42)$$

Bearing in mind that the expression under the integrals in Eq. (42) is invariant under the transformations $\vec{k} \to \vec{k}'$ and $\vec{k} \leftarrow \vec{k}'$, we can rewrite this equation in the form

$$\frac{dS}{dt} = \frac{VK_B}{2} \sum_{\pm} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} W_{\pm}(\vec{k}',\vec{k}) \times \ln\left( \frac{[1+N(\vec{k},t)]N(\vec{k}',t)}{[1+N(\vec{k}',t)]N(\vec{k},t)} \right) [N(\vec{k}',t) - N(\vec{k},t)]. \qquad (43)$$

By the definition $W_{\pm}$ and $N(\vec{k},t)$ in the integrand are positive, and the function

$$F = \ln\left( \frac{1+N(\vec{k},t)}{1+N(\vec{k}',t)} \frac{N(\vec{k}',t)}{N(\vec{k},t)} \right) [N(\vec{k}',t) - N(\vec{k},t)] \qquad (44)$$

is non-negative in any case, i.e., $N(\vec{k}',t) > N(\vec{k},t)$ or reverse. We thus obtain the required result

$$\frac{dS}{dt} \geqslant 0, \qquad (45)$$

expressing the law of increase of the entropy of the photon gas. Note that equality occurs at equilibrium.

## IV. ADIABATIC PHOTON SELF-CAPTURE

In this section, we discuss the phenomenon of photon capture by some potential well. To this end, we consider the distribution of photons in a slowly applied field, which is a function of the density and the relativistic factor of particles, $U = g[n_\alpha(\vec{r},t), \gamma_\alpha(\vec{r},t)]$.

Let $l$ and $\tau$ be the characteristic length and time of variation of the potential. We suppose that

$$\tau \gg \frac{l}{u}. \qquad (46)$$

With this condition in mind, we employ the equation derived in Ref. [25],

$$\vec{k} \cdot \vec{\nabla}_r N(\vec{r},t,\vec{k}) - \frac{\omega_p^2}{2c^2} \vec{\nabla}_r \frac{n_e}{\gamma_e} \cdot \vec{\nabla}_k N(\vec{r},t,\vec{k}) = 0. \qquad (47)$$

In the following, we consider the case when the density and the relativistic factor are functions only of the distance $r$ from a fixed point. Then the solution of Eq. (47) is

$$N(r,k) = n_{0\gamma} f_\gamma[\varepsilon(\vec{k},\vec{r})] = n_{0\gamma} \frac{1}{(2\pi\sigma_0^2)^{3/2}} \exp\left\{ - \frac{k^2 + k_p^2 \delta\rho}{2\sigma_0^2} \right\}, \tag{48}$$

where $\sigma_0$ is the spectral width, and

$$\delta\rho = \frac{\delta n}{n_0 \gamma} + \left( \frac{1}{\gamma} - \frac{1}{\gamma_0} \right), \quad k_p = \frac{\omega_p}{c}. \tag{49}$$

We specifically note here that $\delta\rho$ can be positive as well as negative. Namely, in the case when the density of particles has a cavity, i.e., $\frac{n}{n_0} = 1 - \frac{|\delta n|}{n_0}$, and $\gamma = \gamma_0$, $\delta\rho$ is negative $(\delta\rho = -\frac{1}{\gamma_0} |\delta n| / n_0)$. Next, in the case when the density does not change, i.e., $n = n_0$, but $\gamma > \gamma_0$, i.e., there is a focusing of EM waves, then $\delta\rho$ is again negative. Whereas, in the case when both $n$ and $\gamma$ change, then $\delta\rho$ can be positive as well as negative.

If $\delta\rho < 0$ in some region, and in the rest of the space $\delta\rho > 0$, then we have two sorts of photons. First, photons with $\delta\rho > 0$ have a Gaussian-Boltzmann distribution throughout the space, and the density of photons is given as

$$n_\gamma(r) = \int d\vec{k} N(r,k) = n_{0\gamma} \exp\left\{ - \frac{k_p^2 \delta\rho}{2\sigma_0^2} \right\}; \tag{50}$$

but in the case, when there are some photons in the cavity, then the motion of the photons takes place in a finite region of space, i.e., they are trapped in the potential well $U = -k_p^2 |\delta\rho|$. In other words, for the trapped photons we have $k_0^2 + k_p^2 \delta\rho = 0$, $N(r,k) = N(0)$, and for them the wave number varies between $0 \leqslant k \leqslant k_p |\delta\rho|^{1/2}$, whereas for the untrapped photons, $k > k_p |\delta\rho|^{1/2}$. Therefore we can now represent $n_\gamma$ as

$$n_\gamma(r) = n_\gamma^{trap}(r) + n_\gamma^{untr}(r), \tag{51}$$

where

$$\frac{n_\gamma^{trap}}{n_{0\gamma}} = \frac{4}{3\sqrt{\pi}} \left( \frac{k_p}{\sqrt{2}\sigma_0} |\delta\rho|^{1/2} \right)^3 \tag{52}$$

and for the untrapped photons we have

$$\frac{n_\gamma^{untr}}{n_{0\gamma}} = \frac{4}{\sqrt{\pi}} e^{k_p^2 |\delta\rho| / 2\sigma_0^2} \int_{\eta_0}^{\infty} d\eta \eta^2 e^{-\eta^2}$$

$$= \left( 1 - \frac{4}{\sqrt{\pi}} \int_0^{\eta_0} d\eta \eta^2 e^{-\eta^2} \right) e^{\eta_0^2}, \tag{53}$$

where $\eta = \frac{k}{\sqrt{2}\sigma_0}$ and $\eta_0 = \frac{k_p |\delta\rho|^{1/2}}{\sqrt{2}\sigma_0}$. Equations (52) and (53) exhibit that, when $k_p |\delta\rho|^{1/2} \gg \sqrt{2}\sigma_0$, then $n_\gamma^{untr} \to 0$, whereas $n_\gamma^{trap}$ increases as a third power, i.e., almost all photons are trapped. In the opposite limit, $\eta_0 \ll 1$, for the density of photons, we obtain

$$n_\gamma = n_{0\gamma} \left\{ 1 + \eta_0^2 - \frac{8}{15\sqrt{\pi}} \eta_0^{5/2} \right\}.$$

## V. UNIFORM EXPANSION OF PHOTON GAS

We next consider the uniform expansion of the photon gas. To this end, we employ the equation of continuity of the photon gas derived in Ref. [14]. In the past Kompaneets [23] has shown that the establishment of equilibrium between the photons and the electrons is possible through the Compton effect. In his consideration, since the free electron does not absorb and emit, but only scatters the photon, the total number of photons is conserved. Using the kinetic equation of Kompaneets, Zel'dovich and Levich [24] have shown that in the absence of absorption the photons undergo Bose-Einstein condensation. Recently it was shown that another mechanism exists ("Compton" scattering type) of the creation of equilibrium state and Bose-Einstein condensation in a nonideal dense photon gas [25,14]. Hereafter, we assume that the total number of photons is conserved.

In the following the dynamics of the photon gas is determined by the constancy of the entropy. Equations (30) and (32) yield the following expressions, first for the ultrarelativistic photon gas, i.e., $\varepsilon_\gamma \approx c p_\gamma$,

$$T(t) = T_0 \left( \frac{V_0}{V(t)} \right)^{1/3} \frac{1}{1 + \zeta \left( \frac{V(t)}{V_0} \right)^{1/3}}, \tag{54}$$

and second for the nonrelativistic photon gas, i.e., $\varepsilon_\gamma \approx m_\gamma c^2 + \frac{p_\gamma^2}{2m_\gamma}$,

$$T(t) = T_0 \left( \frac{V_0}{V(t)} \right)^{1/6}, \tag{55}$$

where $\zeta = 0.29 \frac{m_\gamma(V_0) c^2}{T_0}$, $T_0$ and $V_0$ are initial temperature and volume.

In order to determine the explicit dependence $T(t)$ and $V(t)$, we study the spherically symmetric case. In this case the equation of continuity takes the form

$$\frac{\partial n_\gamma}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 n_\gamma u_r = 0. \tag{56}$$

The solution of which we represent as

$$n_\gamma(t) = n_{0\gamma} \left( \frac{R_0}{R(t)} \right)^3, \quad u_r = u_0 \frac{r}{R(t)}, \tag{57}$$

where the suffix 0 denotes the constant initial value.

Substituting Eq. (57) into Eq. (56), we obtain

$$\frac{dR(t)}{dt} = u_0 \quad \text{or } R(t) = R_0 + u_0 t. \tag{58}$$

Substituting Eq. (58) into Eqs. (54) and (55), we can now explicitly express also the time dependence of the temperature. The result is for the ultrarelativistic photon gas

$$T(t) = T_0 \frac{R_0}{R(t)} \frac{1}{1 + \zeta \frac{R(t)}{R_0}} \tag{59}$$

and for the nonrelativistic photon gas

$$T(t) = T_0\left(\frac{R_0}{R(t)}\right)^{1/2}. \tag{60}$$

Thus we may conclude that the cooling of the photon gas is slower in the nonrelativistic case than in the ultrarelativistic case, as is evident from Eqs. (59) and (60).

## VI. SUMMARY

We have investigated the interaction of spectrally broad and relativistically intense EM radiation with a plasma. We have obtained the condition which allows the EM field to be treated as quasiclassical. We have studied the system of a dilute gas composed of electrons, ions, and photons (or electrons, positrons, and photons), and described it in terms of macroscopic properties. We have calculated all thermodynamic quantities developing the statistical mechanics in the presence of a strong EM field. We have demonstrated the existence of the ratio of the specific heats, $\Gamma$, which equals 7/6. We have also disclosed the magnitude of the mean square fluctuation of the number of photons, and shown that the relative fluctuation of the number of photons does not decrease, when the mean number of photons increases. We have discussed the Boltzmann H-theorem in a photon gas. In addition, we have studied the adiabatic photon self-capture and defined the number of trapped photons. Finally, we have considered the uniform expansion of the photon gas and explicitly expressed the time dependence of temperature and volume. EM radiation has played a crucial role in opening up new frontiers in physics. The distribution law discovered by Planck accurately describes the equilibrium properties of an assembly of photons over a vast range of temperatures and scales, from terrestrial cavity radiation to hot stellar atmospheres, and, of course, including the cosmic background radiation. However, there are changes in Planck's law and photon thermodynamics, as discussed in this and previous [5,6] papers, which may play a role in an as yet undiscovered phenomenon.

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